Correlation

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We introduce another product on $\mathcal{F}(\mathbb{R},\mathbb{K})$, the scalar product, from which we define the correlation and relate to convolution.

Definition 0.1 (Scalar product, Hermitian product)
If V denotes a vector space over $\mathbb{R}$, a **scalar product** over V is any mapping $\langle .,. \rangle : V \times V \to \mathbb{R}$ satisfying the following properties:

- it is bilinear: for any $(x,y,z) \in V^3$ and any $(\alpha,\beta) \in \mathbb{R}^2$, $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$; same for the second component;
- it is positive: for any $x \in V$, $\langle x,x\rangle \geq 0$;
- it is definite: for any $x \in V$, $\langle x,x\rangle = 0 \Leftrightarrow x = 0_V$.

If V denotes a vector space over $\mathbb{C}$, a **Hermitian product** over V is any mapping $\langle .,. \rangle : V \times V \to \mathbb{C}$ satisfying the following properties:

- it is linear for the first component: for any $(x,y,z) \in V^3$ and any $(\alpha,\beta) \in \mathbb{C}^2$, $\langle \alpha x + \beta y, z\rangle = \alpha\langle x,z\rangle + \beta\langle y,z\rangle$;
- it is anti-linear for the second component: for any $(x,y,z) \in V^3$ and any $(\alpha,\beta) \in \mathbb{C}^2$, $\langle x, \alpha y + \beta z\rangle = \overline{\alpha}\langle x,y\rangle + \overline{\beta}\langle x,z\rangle$, where $\overline{z}$ denotes the conjugate of $z \in \mathbb{C}$;
- it is positive and definite.

In this lecture, we restrict our study to the subspace $L^2(\mathbb{R},\mathbb{K})$ of $\mathcal{F}(\mathbb{R},\mathbb{K})$ of square-integrable signals.

Definition 0.2 (Scalar product over $\mathbf{L^2}(\mathbb{R},\mathbb{R})$, Hermitian product over $\mathbf{L^2}(\mathbb{R},\mathbb{C})$, energy)
We define a **scalar product** over $L^2(\mathbb{R},\mathbb{R})$ by

$$\forall (x,y) \in L^2(\mathbb{R},\mathbb{R})^2 \qquad \langle x,y\rangle = \int_{-\infty}^{+\infty} x(t)y(t)dt$$

We define a **Hermitian product** over $L^2(\mathbb{R},\mathbb{C})$ by

$$\forall (x,y) \in L^2(\mathbb{R},\mathbb{C})^2 \qquad \langle x,y\rangle = \int_{-\infty}^{+\infty} x(t)\overline{y(t)}dt$$

From these products, we can define the norm of a signal, from which we introduce the **energy**:

$$\forall x \in L^2(\mathbb{R},\mathbb{K}) \qquad E(x) = \|x\|^2 = \langle x,x\rangle$$

i.e.

$$\forall x \in L^2(\mathbb{R},\mathbb{R}) \quad E(x) = \int_{-\infty}^{+\infty} x(t)^2 dt \qquad \forall x \in L^2(\mathbb{R},\mathbb{C}) \quad E(x) = \int_{-\infty}^{+\infty} |x(t)|^2 dt$$

Remarks:

- In other words, $L^2(\mathbb{R}, \mathbb{K})$ is the subspace of $\mathcal{F}(\mathbb{R}, \mathbb{K})$ of finite-energy signals.
- Cauchy-Schwarz inequality indicates that for two signals x and y of $L^2(\mathbb{R}, \mathbb{K})$, $|\langle x, y \rangle| \leq \|x\|.\|y\| = \sqrt{E(x)E(y)}$, ensuring that both products are well defined over $L^2(\mathbb{R}, \mathbb{K})$.
- Notation $\langle ., . \rangle$ for the scalar product is consistent with the duality bracket, since for any fixed signal y, mapping $x \mapsto \langle x, y \rangle$ is a linear form.
- To deal indistinctly with both products, we use notation x^* to designate $x^* = x$ for $x \in \mathbb{R}$, and $x^* = \overline{x}$ for $x \in \mathbb{C}$.
- For infinite-energy signals, we can introduce the notion of average power.

Definition 0.3 (Average power)
The **average power** of a signal $x \in \mathcal{F}(\mathbb{R}, \mathbb{K})$ is the real number:

$$P(x) = \lim_{t \to +\infty} \frac{1}{2t} \int_{-t}^{t} |x(u)|^2 du$$

Remark: Finite-energy signals have a zero average power.

Definition 0.4 (Cross-correlation, autocorrelation)
Let x and y be two signals of $L^2(\mathbb{R}, \mathbb{K})$. The **cross-correlation** is the function $\gamma_{xy} : \mathbb{R} \to \mathbb{K}$ defined by

$$\forall t \in \mathbb{R} \qquad \gamma_{xy}(t) = \langle x, \tau_t(y) \rangle = \int_{-\infty}^{+\infty} x(u) y^*(u - t) du$$

The **autocorrélation** of a signal $x \in L^2(\mathbb{R}, \mathbb{K})$ is the cross-correlation with itself, i.e.

$$\forall t \in \mathbb{R} \qquad \gamma_x(t) = \gamma_{xx}(t) = \langle x, \tau_t(x) \rangle = \int_{-\infty}^{+\infty} x(u) x^*(u - t) du$$

Remarks:

- As a scalar product, cross-correlation measures the similarity between a signal x and a shifted version of a signal y. It enables the identification of common "patterns" between two signals. Autocorrelation enables the identification of similarities between a signal x and a shifted version of itself, which can be used to determine the periodicity of the signal for instance.
- For any signal $x \in \mathcal{F}(\mathbb{R}, \mathbb{K})$, $\gamma_x(0) = \langle x, x \rangle = E(x)$, thus the energy of a signal is equal to its autocorrelation in 0.
- The convolution can be seen as a variant of cross-correlation. Indeed, let x and y be two signals of $\mathcal{F}(\mathbb{R}, \mathbb{K})$. For any $t \in \mathbb{R}$,

$$(x * y)(t) = \int_{-\infty}^{+\infty} x(u) y(t - u) du = \int_{-\infty}^{+\infty} x(u) \tilde{y}(u - t) du = \langle x, \tau_t(\tilde{y})^* \rangle = \gamma_{x\tilde{y}^*}(t)$$

with $\tilde{y} : t \mapsto y(-t)$. Conversely, we can write cross-correlation as a function of convolution: $\gamma_{xy} = x * \tilde{y}^*$.

- By connecting convolution to this scalar product, we can bring another proof that any LTI system is a convolution system. Indeed, let L be an LTI system of impulse response $h = L(\delta)$, $x \in \mathcal{F}(\mathbb{R}, \mathbb{K})$ an input and $y = L(x)$ the corresponding output. For any $t \in \mathbb{R}$,

$$y(t) = L(x)(t) = L(x * \delta)(t) = L\left(\langle x, \tau_t(\tilde{\delta})\rangle\right) = \langle x, L(\tau_t(\tilde{\delta}))\rangle = \langle x, \tau_t(\widetilde{L(\delta)})\rangle = \langle x, \tau_t(\tilde{h})\rangle = (x * h)(t)$$

where we use the bilinearity of L and its commutativity with operators τ_t and $x \mapsto \tilde{x}$.

Proposition 0.1
We have the following properties:

(i) For any two signals x and y, cross-correlation satisfies the equality

$$\forall t \in \mathbb{R} \qquad |\gamma_{xy}(t)| \leq \sqrt{E(x)E(y)}$$

In particular, for any signal x, the absolute value of autocorrelation γ_x reaches its maximum $E(x)$ in 0.

(ii) Autocorrelation satisfies the following symmetry property: for any signal x, for any $t \in \mathbb{R}$, $\gamma_x(-t) = \gamma_x^*(t)$.

Proof : (i) First note that any shifted signal $\tau_t(x)$ has the same energy as signal x. Indeed, by the change of variable $u \mapsto u + t$, we get

$$\forall t \in \mathbb{R} \qquad E(\tau_t(x)) = \int_{-\infty}^{+\infty} |\tau_t(x)(u)|^2 du = \int_{-\infty}^{+\infty} |x(u-t)|^2 du = \int_{-\infty}^{+\infty} |x(u)|^2 du = E(x)$$

Then by applying Cauchy-Schwarz inequality,

$$\forall t \in \mathbb{R} \qquad |\gamma_{xy}(t)| = |\langle x, \tau_t(y)\rangle| \leq \|x\|.\|\tau_t(y)\| = \sqrt{E(x)E(\tau_t(y))} = \sqrt{E(x)E(y)}$$

In the particular case of $y = x$,

$$\forall t \in \mathbb{R} \qquad |\gamma_x(t)| \leq E(x) = \gamma_x(0)$$

(ii) Let a signal x and $t \in \mathbb{R}$. By the change of variable $u \mapsto u - t$, we get:

$$\gamma_x(-t) = \int_{-\infty}^{\infty} x(u)x^*(u+t)du = \int_{-\infty}^{\infty} x(u-t)x^*(u)du = \left(\int_{-\infty}^{\infty} x(u)x^*(u-t)du\right)^* = \gamma_x^*(t) \qquad \blacksquare$$

Remark: It is consistent that the maximum of autocorrelation is in 0, since a signal has a maximum of similarity with a version of itself shifted by 0.